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ON INTRINSIC ERGODICITY OF PIECEWISE MONOTONIC TRANSFORMATIONS WITH POSITIVE ENTROPY II

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ABSTRACT
The results about measures with maximal entropy, which are proved in [3], are extended to the following more general class of transformations on the unit interval $I: I = \bigcup_{i=1}^{n} J_i$, the J_i are disjoint intervals, f/J_i is increasing or decreasing and continuous, and $h_{top}(f) > 0$.

§0. Introduction

The aim of this paper is to extend the results of [3] to a more general class of dynamical systems. We consider (I, f), where $I = [0,1] = \bigcup_{i=1}^{n} J_i$, the J_i are disjoint intervals and f/J_i is continuous and either strictly increasing or strictly decreasing. Again we assume

(a) $(J_1, J_2, \cdots, J_n)$ is a generator for (I, f),

(b) $h_{top}(f) > 0$.

An invariant measure μ on (I, f) is called maximal, if its entropy $h(\mu)$ is greater than or equal to the entropy of every other invariant measure, or, what is the same by the variational principle [2, §18], if $h(\mu)$ is equal to the topological entropy $h_{top}(f)$ of (I, f). We want to get the same results about maximal measures on (I, f) as in [3] for piecewise increasing transformations.

The f-expansion φ (see §1 for definition) gives an isomorphism of (I, f) with a shift space Σ_f^+, which preserves maximal measures. But φ is not order preserving, as it was in [3]. Lemma 4 of [3] does not hold and all results using this lemma need a new proof. To this end we construct a piecewise increasing transformation g on I, such that (I, f) is a factor of (I, g), and move the problems to (I, g). And for this case they are solved in [3]. Hence all results proved there are also

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valid in this more general case (Theorems 1 and 2 of §2 and Theorems 3 and 4 of §3).

§1. Let (I, f) be as in the introduction. Σ_n^+ denotes the full one-sided shift over $\{1, 2, \cdots, n\}$, σ the shift transformation and $\leqq$ the lexicographic ordering on Σ_n^+. The f-expansion $\varphi: I \to \Sigma_n^+$ is defined by $\varphi(x) = \mathbf{x} = x_0 x_1 x_2 \cdots$, such that $f^i(x) \in J_{x_i}$ for $i \geqq 0$. We have $\varphi \circ f = \sigma \circ \varphi$. Furthermore φ is injective because of (a), but not order preserving (unless f/J_i is increasing for all i). Call i good, if f/J_i is increasing and bad, if f/J_i is decreasing. We introduce a second order relation $\lhd$ in Σ_n^+. Let $\mathbf{x}, \mathbf{y} \in \Sigma_n^+$ and $k \geqq 0$ be the largest integer such that $x_i = y_i$ for $0 \leqq i \leqq k-1$. $\mathbf{x} \lhd \mathbf{y}$, if $\mathbf{x} = \mathbf{y}$ or if the number of bad x_i's in $x_0 x_1 \cdots x_{k-1}$ is even and $x_k < y_k$ or if the number of bad x_i's in $x_0 x_1 \cdots x_{k-1}$ is odd and $x_k > y_k$. With this definition we have $x \leqq y$ in I if and only if $\varphi(x) \lhd \varphi(y)$ in Σ_n^+. Define $\Sigma_f^+ = \overline{\varphi(I)} \subset \Sigma_n$. $\Sigma_f^+ \setminus \varphi(I)$ is countable (the proof is the same as that of lemma 2 of [3]), hence a nullset for every maximal measure. Therefore (I, f) and Σ_f^+ have isomorphic sets of measures with maximal entropy. If $J_i = (r, s)$ set $\mathbf{a}^i = \lim_{t \downarrow r} \varphi(t)$ and $\mathbf{b}^i = \lim_{t \uparrow s} \varphi(t)$. The proof of the following result is as in [3] (cf. also §9 of [1]).

$$\Sigma_f^+ = \{\mathbf{x} \in \Sigma_n^+ : \mathbf{a}^{x_k} \lhd \sigma^k \mathbf{x} \lhd \mathbf{b}^{x_k} \quad \text{for all } k \geqq 0\}.$$

Now we construct a piecewise increasing transformation $g: I \to I$, of which (I, f) is a factor, and move the problems to (I, g). Set $K_i = \{x/2 : x \in J_i\}$ and $K_{i'} = \{1 - x/2 : x \in J_i\}$ $(1 \leqq i \leqq n)$, where i' stands here and in the sequel for $2n - i + 1$. Write f_i for f/J_i. Define $g_i : K_i \to I$ by

$$g_i(x) = \begin{cases} f_i(2x)/2 & \text{if } i \text{ is good} \\ 1 - f_i(2x)/2 & \text{if } i \text{ is bad} \end{cases} \qquad \text{for } 1 \leqq i \leqq n,$$

$$g_i(x) = 1 - g_{i'}(1 - x) \qquad \text{for } n + 1 \leqq i \leqq 2n$$

and $g(x) = g_i(x)$, if $x \in K_i$ $(1 \leqq i \leqq 2n)$. For $x = \frac{1}{2}$ choose one of the two possibilities. Figure 1 shows an example of an f and of the corresponding g. g has the property that

$$1 - g(x) = g(1 - x). \tag{1.1}$$

Define $p: (I, g) \to (I, f)$ by

$$p(x) = \begin{cases} 2x & \text{if } x \leqq \frac{1}{2}, \\ 2 - 2x & \text{if } x \geqq \frac{1}{2}. \end{cases}$$

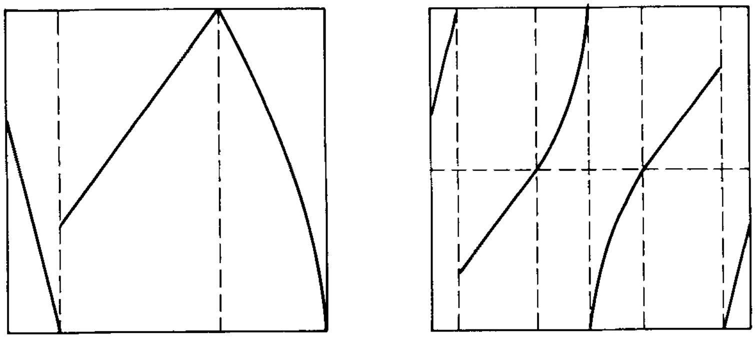


Fig. 1.

Then $p \circ g = f \circ p$ and p is 2-1 (except for the point $x = \frac{1}{2}$). Let $\tilde{\varphi} : (I, g) \to \Sigma_{2n}^{+}$ denote the g-expansion and, if $K_j = (r, s)$, set $c^j = \lim_{t \downarrow r} \tilde{\varphi}(t)$ and $d^j = \lim_{t \uparrow s} \tilde{\varphi}(t)$ $(1 \leqq j \leqq 2n)$. We have by (1.1)

$$(1.2) \qquad (c_k^j)' = d_k^{j'} \quad \text{for } 1 \leqq j \leqq 2n \quad \text{and} \quad k \geqq 0.$$

From [3] we know that $\overline{\tilde{\varphi}(I)} = \Sigma_g^+ = \{ y \in \Sigma_{2n}^+ : c^{y_k} \leqq \sigma^k y \leqq d^{y_k} \ \forall k \geqq 0 \}$. Let Σ_f and Σ_g denote the natural extensions of Σ_f^+ and Σ_g^+ respectively ($\Sigma_f = \{ x \in \Sigma_n = \{1, \cdots, n\}^Z : x_k x_{k+1} \cdots \in \Sigma_f^+ \ \forall k \in Z \}$ and there is a 1-1 correspondence between the set of invariant measures of a shift space and that of its natural extension preserving the entropy). We consider $\varphi \circ p \circ \tilde{\varphi}^{-1} : \Sigma_g^+ \to \Sigma_f^+$ and denote it again by p.

LEMMA 1. *$p : \Sigma_g^+ \to \Sigma_f^+$ is given by $p(y) = q(y_0) q(y_1) \cdots$, where $q(i) = i$, if $1 \leqq i \leqq n$ and i', if $n + 1 \leqq i \leqq 2n$.*

The proof is evident. We have also $p : \Sigma_g \to \Sigma_f$ given by $p(y) = \cdots q(y_{-1}) q(y_0) q(y_1) \cdots$. Furthermore

$$(1.3) \qquad p(c^i) = p(d^{i'}) = \begin{cases} a^i & \text{for } 1 \leqq i \leqq n, \\ b^{i'} & \text{for } n+1 \leqq i \leqq 2n, \end{cases}$$

$$(1.4) \qquad p^{-1}(x) = \{ y, y' \}, \qquad \text{where } y_i = \begin{cases} x_i & \text{if } g(K_{y_{i-1}}) \subset [0, \frac{1}{2}], \\ x_i' & \text{if } g(K_{y_{i-1}}) \subset [\frac{1}{2}, 1] \end{cases}$$

(y' stands for $y_0' y_1' \cdots$ or $\cdots y_{-1}' y_0' y_1' \cdots$).

From (1.4) one sees easily that Σ_g^+ contains exactly twice as many admissible blocks (i.e. blocks occurring in points of Σ_g^+) of length n as Σ_f^+. Hence it follows directly from the formula for h_{top} that $h_{\text{top}}(\Sigma_g^+) = h_{\text{top}}(\Sigma_f^+)$.

§2. Let ${}_0[x_0x_1\cdots x_{m-1}]=\{y\in\Sigma_f^+: y_i=x_i \text{ for } 0\leqq i\leqq m-1\}$ denote a cylinder set. For an admissible block $x_0x_1\cdots x_{m-1}$ in Σ_f^+, i.e. ${}_0[x_0\cdots x_{m-1}]\neq\emptyset$, define $G_{x_0\cdots x_{m-1}}=\sigma^m({}_0[x_0\cdots x_{m-1}])$. Let $D=\{(x_{m-1},G_{x_0\cdots x_{m-1}}): {}_0[x_0\cdots x_{m-1}]\neq\emptyset\}$. Together with the arrows $(x_{m-1},G_{x_0\cdots x_{m-1}})\rightarrow(x_m,G_{x_0\cdots x_m})$, D becomes a diagram M (cf. [3]). Set $\Sigma_M=\{z\in D^{\mathbf{Z}}$: there is an arrow from z_i to z_{i+1} in M for all $i\in\mathbf{Z}\}$. As in [3] we want to define an isomorphism $\psi:\Sigma_M\backslash N\rightarrow\Sigma_M$, where $N=\{x\in\Sigma_f:\exists m\in\mathbf{Z}$ so that $\forall j<m\ \exists k\leqq j$ with $x_kx_{k+1}\cdots x_m=a_0^i\cdots a_{m-k}^i$ or $b_0^i\cdots b_{m-k}^i$ for some i and $\cdots x_{m-2}x_{m-1}x_m$ is not periodic$\}$.

To do this set $H_{y_0\cdots y_{m-1}}=\sigma^m({}_0[y_0\cdots y_{m-1}])\subset\Sigma_g^+$ for all admissible blocks in Σ_g^+, set $\tilde{D}=\{(y_{m-1},H_{y_0\cdots y_{m-1}}): {}_0[y_0\cdots y_{m-1}]\neq\emptyset$ in $\Sigma_g^+\}$ and together with the arrows $(y_{m-1},H_{y_0\cdots y_{m-1}})\rightarrow(y_m,H_{y_0\cdots y_m})$ we get a diagram $\tilde{M}$. Define $r:\tilde{D}\rightarrow D$ by

$$r(y_{m-1},H_{y_0\cdots y_{m-1}})=(q(y_{m-1}),G_{q(y_0)\cdots q(y_{m-1})}).$$

If $z_1,z_2\in\tilde{D}$, there is an arrow $z_1\rightarrow z_2$ if and only if there is an arrow $r(z_1)\rightarrow r(z_2)$, i.e. r respects the arrows in $\tilde{M}$ and M.

Let $\Sigma_{\tilde{M}}$ be defined as Σ_M. Define $p:\Sigma_{\tilde{M}}\rightarrow\Sigma_M$ by

$$p(\cdots z_{-1}z_0z_1\cdots)=\cdots r(z_{-1})r(z_0)r(z_1)\cdots.$$

The sequence at the right hand side is in Σ_M, because r respects the arrows of $\tilde{M}$ and M. r and p are again 2-1. As g is piecewise increasing we have an isomorphism $\tilde{\psi}:\Sigma_g\backslash\tilde{N}\rightarrow\Sigma_{\tilde{M}}$ constructed in §2 of [3], where $\tilde{N}=\{y\in\Sigma_g:\exists m\in\mathbf{Z}$ so that $\forall j<m\ \exists k\leqq j$ with $y_k\cdots y_m=c_0^i\cdots c_{m-k}^i$ or $d_0^i\cdots d_{m-k}^i$ for some i and $\cdots y_{m-1}y_m$ is not periodic$\}=p^{-1}(N)$. We define $\psi:\Sigma_f\backslash N\rightarrow\Sigma_M$ by $\psi=p\circ\tilde{\psi}\circ p^{-1}$, i.e. such that the following diagram is commutative.

$$\begin{array}{ccc}\Sigma_g\backslash\tilde{N} & \xrightarrow{\tilde{\psi}} & \Sigma_{\tilde{M}}\\ \downarrow p & & \downarrow p\\ \Sigma_f\backslash N & \xrightarrow{\psi} & \Sigma_M\end{array}$$

LEMMA 2. *ψ is well defined and an isomorphism.*

PROOF. Let $x\in\Sigma_f\backslash N$. $p^{-1}(x)$ contains two points y and y' (cf. (1.4)). But $\tilde{\psi}(y)$ and $\tilde{\psi}(y')$ are mapped to the same $z\in\Sigma_M$ by p because, if $\tilde{\psi}(y)$ has entry $(y_m,G_{y_i\cdots y_m})$ at some coordinate, then $\tilde{\psi}(y')$ has entry $(y'_m,G_{y'_i\cdots y'_m})$ at this coordinate (cf. the definition of $\tilde{\psi}$ in §2 of [3]). That ψ is an isomorphism, follows because $\tilde{\psi}$ is one. One can define ψ^{-1} in the same way as ψ.

LEMMA 3. *If ν is a σ-invariant measure concentrated on N then the entropy $h(\nu)$ of ν is zero.*

PROOF. Let $\tilde{\nu}$ be the measure on Σ_g defined by

$$\tilde{\nu}({}_0[y_0\cdots y_{m-1}])=\tfrac{1}{2}\nu({}_0[q(y_0)\cdots q(y_{m-1})]).$$

It is σ-invariant, because $k\in\{1,2,\cdots,2n\}$ can be preceded either by m or by $m'\in\{1,2,\cdots,2n\}$ in a point $y\in\Sigma_g$, not by both (cf. the definition of g). It follows directly from the formula for the entropy that $h(\tilde{\nu})=h(\nu)$. Furthermore $\tilde{\nu}\circ p^{-1}=\nu$ and $p^{-1}(N)=\tilde{N}$, hence $\tilde{\nu}(\tilde{N})=1$. By lemma 7 of [3] it follows that $h(\tilde{\nu})=0$ and hence $h(\nu)=0$.

Because of (b) it follows from Lemma 3 that N is a nullset for every maximal measure (cf. §0 of [3]). Hence

THEOREM 1. *(Σ_f,σ) and (Σ_M,σ) have isomorphic sets of maximal measures, and hence the same is true for (I,f) and (Σ_M,σ).*

We consider M as a matrix with entries 0 and 1. $M_{jk}=1$, if and only if we have an arrow $j\to k$ $(j,k\in D)$. Let $M^1,M^2,\cdots$ be the irreducible submatrices of M. The proof of the following theorem about Σ_M is the same as in [3].

THEOREM 2. (i) *$h_{\text{top}}(\Sigma_M)=\log r(M)$, the logarithm of the spectral radius of the l^1-operator $u\mapsto uM$ $(u\in l^1)$.*

(ii) *Every ergodic maximal measure is concentrated on a $\Sigma_{M^i}=\{z\in D^{\mathbf{Z}}:M^i_{z_kz_{k+1}}=1$ for all $k\in\mathbf{Z}\}$ satisfying $r(M^i)=r(M)$.*

(iii) *For every such M^i there is at most one maximal measure concentrated on Σ_{M^i}. It is Markov given by $\pi_j=u_jv_j$ and $P_{jk}=M^i_{jk}v_k/r(M)v_j$ $(j,k\in D)$, where u is a left and v a right eigenvector of M^i for the eigenvalue $r(M^i)=r(M)$.*

It suffices to determine the ergodic maximal measures, because they are the extremal points of the compact convex set of all maximal measures. For this one can use Theorem 2. To get more information about M, we can use the results about $\tilde{M}$ obtained in [3]. As there we divide every $\boldsymbol{c}^i$ $(1\leqq i\leqq 2n)$ into initial segments of the $\boldsymbol{d}^j$ $(1\leqq j\leqq 2n)$ and denote their lengths by $r(i,1)$, $r(i,2),\cdots$ $(r(i,m)\geqq 1)$, i.e.

$$c^i_{r(i,1)+\cdots+r(i,m)+k}=d^j_k \qquad \text{for } 0\leqq k\leqq r(i,m+1)-1, \quad j=c^i_{r(i,1)+\cdots+r(i,m)}.$$

$$c^i_{r(i,1)+\cdots+r(i,m+1)}\neq d^j_{r(i,m+1)}$$

Similarly divide every $\boldsymbol{d}^j$ $(1\leqq j\leqq 2n)$ into initial segments of the $\boldsymbol{c}^i$ $(1\leqq i\leqq 2n)$ and denote their lengths by $s(j,1),s(j,2),\cdots$ $(s(j,m)\geqq 1)$. We have always $r(i,1)=s(i,1)$ and by (1.2) we have $r(i,m)=s(i',m)$ for $1\leqq i\leqq 2n$ and $m\geqq 1$. One gets $\boldsymbol{c}^i$ and $\boldsymbol{d}^i$ as $p^{-1}(\boldsymbol{a}^i)=\{\boldsymbol{c}^i,\boldsymbol{d}^{i'}\}$ and $p^{-1}(\boldsymbol{b}^i)=\{\boldsymbol{c}^{i'},\boldsymbol{d}^i\}$ (cf. (1.3)). In [3] it is proved that $\tilde{M}$ can be described as follows. $\tilde{D} = \{(A, i, k), (B, j, k): 1 \leqq i, j \leqq 2n, k \geqq 1$ and $(A, i, k) = (B, i, k)$ for $1 \leqq k \leqq r(i, 1)\}$. $\tilde{M}$ has the following arrows.

$(A, i, k) \to (A, i, k+1)$, $(B, j, k) \to (B, j, k+1)$.

If $k = r(i,1) + \cdots + r(i,m)$ for some m then $(A, i, k) \to (B, j, r(i,m)+1)$, where $j = c^i_{r(i,1)+\cdots+r(i,m-1)}$ and $(A, i, k) \to (A, t, 1) = (B, t, 1)$ for $c^i_k < t < d^i_{r(i,m)}$.

If $k = s(j,1) + \cdots + s(j,m)$ for some m then $(B, j, k) \to (A, i, s(j,m)+1)$, where $i = d^j_{s(j,1)+\cdots+s(j,m-1)}$ and $(B, j, k) \to (A, t, 1)) = (B, t, 1)$ for $c^i_{s(j,m)} < t < d^j_k$.

One gets M applying r to the elements of $\tilde{D}$. As (A, i, k) stands for $(c^i_{k-1}, H_{c^i_0 \cdots c^i_{k-1}})$ and (B, j, k) for $(d^j_{k-1}, H_{d^j_0 \cdots d^j_{k-1}})$ we have $r(A, i, k) = r(B, i', k)$. Hence we get M identifying (A, i, k) and (B, i', k) in $\tilde{D}$. Denote this equivalence relation by $\sim$. Then $M = \tilde{M}/\sim$. Arrows are respected by $\sim$, because they are respected by r.

Example. Let $n = 2$, f/J_1 is increasing, $f(\text{end point of } J_1) = 1$, f/J_2 is decreasing and $f(J_2) = I$, i.e. f has a graph like the one in Fig. 2. At the right hand side we have drawn the graph of the corresponding g.

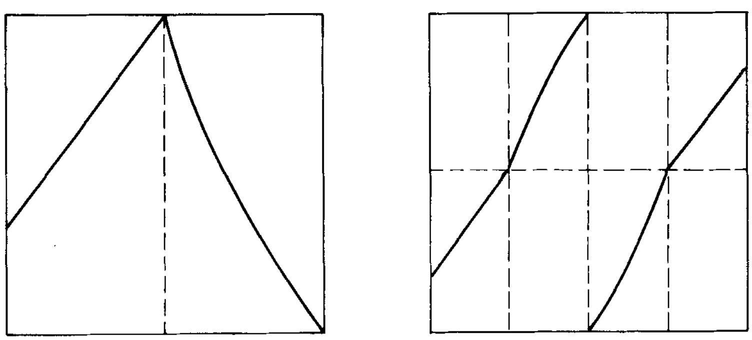
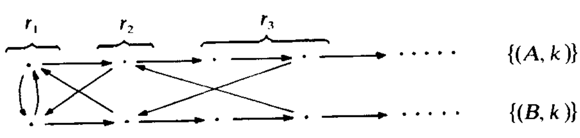
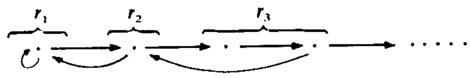


Fig. 2.

From [4] we know what the diagram $\tilde{M}$ for g looks like. Let $\boldsymbol{c}$ be the g-expansion of 0 and $\boldsymbol{d}$ the g-expansion of 1. Let $r_1, r_2, \cdots$ be the lengths of initial segments of $1\boldsymbol{d}$ in $\boldsymbol{c}$. By symmetry of g, $r_1, r_2, \cdots$ are also the lengths of initial segments of $2\boldsymbol{c}$ in $\boldsymbol{d}$. $\tilde{D}$ can be represented by $\{(A, k), (B, k): k \geqq 1\}$ and we have arrows $(A, k) \to (A, k+1)$, $(B, k) \to (B, k+1)$, $(A, r_1 + \cdots + r_m) \to (B, r_m)$, $(B, r_1 + \cdots + r_m) \to (A, r_m)$ (cf. §1 of [4]).



We get M identifying (A, k) and (B, k). Hence $D = \{k : k \geqq 1\}$ and we have arrows $k \to k+1$ and $r_1 + \cdots + r_m \to r_m$



We know from [4] that $\tilde{M}$ contains exactly one irreducible submatrix with maximal spectral radius. Hence the same is true for M and by Theorem 2 an (I, f) as above has always unique maximal measure.

§3. We want to generalize two theorems of [3].

THEOREM 3. (i) $I = \bigcup I_i$ *(finite or countable union), where every* I_i *is a finite union of closed intervals. If* $i < j$, I_i *and* I_j *have at most finitely many points in common and there is no* $x \in I_j \setminus I_i$ *with* $f^k(x) \in I_i$ *for some* $k \geqq 0$.

(ii) *There are only finitely many ergodic maximal measures on* (I, f). *Their supports are* $\Omega_i = \bigcap_{k=0}^{\infty} f^{-k}(I_i)$ *(different i's for different ergodic maximal measures). Either* $\Omega_i = I_i$ *or* Ω_i *is a Cantor-like set.*

PROOF. We know from [3] that Theorem 3 is valid for (I, g). Hence there are $\tilde{I}_i$, which satisfy (i) for g instead of f. Every $\tilde{I}_i$ corresponds to an irreducible submatrix of $\tilde{M}$. The map $(y_{k-1}, H_{y_0 \cdots y_{k-1}}) \mapsto (y'_{k-1}, H_{y'_0 \cdots y'_{k-1}})$ transforms irreducible submatrices of $\tilde{M}$ into irreducible submatrices of $\tilde{M}$. Hence it follows that the map $x \mapsto 1 - x$ transforms every $\tilde{I}_i$ into an $\tilde{I}_j$ (it may be that $j = i$) (cf. [3]). Hence we have

$$p / \tilde{I}_i \text{ is either 1-1 or 2-1.} \tag{3.1}$$

Take the sets $p(\tilde{I}_i)$ as the I_i and (i) is satisfied for (I, f). To prove (ii) let μ be an ergodic maximal measure of (I, f) and define $\tilde{\mu}$ on (I, g) by $\tilde{\mu}(A) = \frac{1}{2}\mu(p(A))$, where A is a measurable subset of $[0, \frac{1}{2}]$ or $[\frac{1}{2}, 1]$. This is the same definition as in the proof of Lemma 3 (we have used the isomorphic shift spaces there). For the same reason as there, $\tilde{\mu}$ is an invariant measure and has the same entropy as μ. Because of $h_{\mathrm{top}}(f) = h_{\mathrm{top}}(g)$, $\tilde{\mu}$ is a maximal measure for (I, g). $\tilde{\mu}$ can be written as a linear combination of the finitely many ergodic maximal measures of (I, g). Hence the support of $\tilde{\mu}$ is a finite union of $\tilde{\Omega}_i = \bigcap_{k=0}^{\infty} g^{-k}(\tilde{I}_i)$. For such an $\tilde{\Omega}_i$, we have $f^{-1}(p(\tilde{\Omega}_i)) \subset p(\tilde{\Omega}_i)$ and $\mu(p(\tilde{\Omega}_i)) > 0$. Hence by ergodicity, $\mu(p(\tilde{\Omega}_i)) = 1$. As $p(\tilde{\Omega}_i)$ is closed we have $\operatorname{supp} \mu = p(\tilde{\Omega}_i)$. Since p is 2-1 and two of the $\tilde{\Omega}_i$ have at most finitely many points in common, it follows that either $\operatorname{supp} \tilde{\mu} = \tilde{\Omega}_i$ for some i, i.e. $\tilde{\mu} =: \nu$ itself is ergodic, or $\operatorname{supp} \tilde{\mu} = \tilde{\Omega}_i \cup \tilde{\Omega}_j$ for $i \neq j$ such that $1 - \tilde{\Omega}_i = \tilde{\Omega}_j$, i.e. $\tilde{\mu} = \tilde{\mu}/\tilde{\Omega}_i + \tilde{\mu}/\tilde{\Omega}_j$ and both $2\tilde{\mu}/\tilde{\Omega}_i =: \nu$ and $2\tilde{\mu}/\tilde{\Omega}_j$ are ergodic. Hence in any case we have found an ergodic maximal measure ν on (I, g) satisfying $\mu = \nu \circ p^{-1}$. Hence (I, f) has only finitely many ergodic maximal measures and $\operatorname{supp} \mu = p(\bigcap_{k=0}^{\infty} g^{-k}(\tilde{I}_i)) = \bigcap_{k=0}^{\infty} f^{-k}(p(\tilde{I}_i)) = \bigcap_{k=0}^{\infty} f^{-k}(I_i)$ (use (3.1)).

THEOREM 4. *If (I, f) is topologically transitive, i.e. there is a dense orbit, then (I, f) has unique maximal measure, which is positive on every nonempty open subset of I.*

PROOF. For some i, I_i of (i) of Theorem 3 must be equal to I and I_j for $j \neq i$ is a finite set. Otherwise there cannot be a dense orbit. If μ is an ergodic maximal measure (at least one such measure must exist, because Σ_f^+ is expansive), $\operatorname{supp} \mu = \bigcap_{k=0}^{\infty} f^{-k}(I_i) = I$ by Theorem 3 (a measure concentrated on I_j for $j \neq i$ has zero entropy). Hence μ is the unique maximal measure and $\operatorname{supp} \mu = I$.

§4. We want to show that all results of this paper remain valid without the assumption

(a) $(J_1, J_2, \cdots, J_n)$ is a generator for f.

This was used only once throughout the paper, namely to show that φ is injective. If (a) is not satisfied, φ is not injective. Let $x, y \in I$ and suppose that $\varphi(x) = \varphi(y) = \boldsymbol{x}$, i.e. $f^k(x)$ and $f^k(y)$ are in the same J_i for every $k \geqq 0$. If z is in the interval with endpoints x and y, it follows that $f^k(z)$ is in the interval with endpoints $f^k(x)$ and $f^k(y)$. Hence $\varphi(z) = \boldsymbol{x}$. This means that $\varphi^{-1}(\boldsymbol{x})$ is a subinterval of I. Let $H = \{\boldsymbol{x} \in \Sigma_f^+ : \varphi^{-1}(\boldsymbol{x}) \text{ is an interval}\}$. We have $\sigma(H) \subset H$. As there can be only countably many disjoint subintervals of I with positive length, it follows that H is at most countable. We show that $\varphi^{-1}(H)$ is a nullset for every maximal measure μ. It suffices to show this for $\varphi^{-1}(\boldsymbol{x})$ for all $\boldsymbol{x} \in H$. We consider three cases.

(i) None of the points $\sigma^k(\boldsymbol{x})$, $k \geqq 0$ is periodic. Then $\varphi^{-1}(\boldsymbol{x})$, $\varphi^{-1}(\sigma\boldsymbol{x}), \cdots$ are disjoint and have increasing measure. As $\mu(I) = 1$ it follows that $\mu(\varphi^{-1}(\boldsymbol{x})) = 0$.

(ii) $\boldsymbol{x}$ is periodic, i.e. there is a p such that $\sigma^p(\boldsymbol{x}) = \boldsymbol{x}$. Then $\varphi^{-1}(\sigma^p(\boldsymbol{x})) \subset \varphi^{-1}(\boldsymbol{x})$ and $f^p/\varphi^{-1}(\boldsymbol{x}) : \varphi^{-1}(\boldsymbol{x}) \to \varphi^{-1}(\boldsymbol{x})$ is monotone. If ν is an f-invariant measure concentrated on $\varphi^{-1}(\boldsymbol{x})$, one has $h(\nu) = 0$. As $h_{top}(f) > 0$, $\varphi^{-1}(\boldsymbol{x})$ is a nullset for every maximal measure.

(iii) $\boldsymbol{x}$ is not periodic, but there is an m such that $\sigma^m(\boldsymbol{x})$ is periodic. Let $K = \bigcup_{k \geqq m} \varphi^{-1}(\sigma^k(\boldsymbol{x}))$. Then $K \subset f^{-1}(K)$ and $\varphi^{-1}(\boldsymbol{x}) \subset f^{-m}(K) \backslash K$. Hence $\mu(\varphi^{-1}(\boldsymbol{x})) \leqq \mu(f^{-m}(K)) - \mu(K) = \mu(K) - \mu(K) = 0$.

In [3] μ is brought back from Σ_M first to Σ_f^+. This is not affected by the nonvalidity of (a). It is easy to bring μ back to I, because $\varphi : \overline{I\backslash\varphi^{-1}(H)} \to \Sigma_f^+\backslash C$ is bijective for some countable set C. We get all maximal measures of (I, f), because there is no maximal measure concentrated on the invariant set $\varphi^{-1}(H)$. The proofs in §3 work unchanged.

As an example consider

$$f(x) = \begin{cases} 3x & \text{for } 0 \leqq x \leqq \frac{1}{3}, \\ x & \text{for } \frac{1}{3} < x < \frac{2}{3}, \\ 3x - 2 & \text{for } \frac{2}{3} \leqq x \leqq 1. \end{cases}$$

Then $\overline{I\backslash\varphi^{-1}(H)}$ is the Cantor set of all x, whose triadic expansion contains only 0 and 2.

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